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Solving internal habit formation models through dynamic programming in infinite dimension

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Solving Internal Habit Formation Models through Dynamic Programming in Infinite Dimension

Emmanuelle Augeraud-Veron · Mauro Bambi · Fausto Gozzi

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Abstract In this paper, we study an economic model where internal habits play a role. Their formation is described by a more general functional form than is usually assumed in the literature, because a finite memory effect is allowed. Indeed, the problem becomes the optimal control of a standard ordinary

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differential equation, with the past of the control entering both the objective function and an inequality constraint. Therefore, the problem is intrinsically infinite dimensional. To solve this model, we apply the dynamic programming approach and we find an explicit solution for the associated Hamilton-Jacobi-Bellman equation, which lets us write the optimal strategies in feedback form. Therefore, we contribute to the existing literature in two ways. Firstly, we fully develop the dynamic programming approach to a type of problem not studied in previous contributions. Secondly, we use this result to unveil the global dynamics of an economy characterized by generic internal habits.

Keywords Optimal Control Problems with Delay · Dynamic Programming · Habit Formation.

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1 Introduction

Motivation – In the past decades, many contributions in economics and finance have studied optimal control problems, where the utility function of the optimizing agent depends, not only on consumption, but also on habits. Habits have been introduced to capture how the utility or satisfaction of an individual may depend, not only on actual consumption, but also on the comparison with past consumption. Since the habits are formed over the agent's own past consumption, they are often called *internal habits*.[1]

[1] Other contributions to the literature have studied *external habits*, i.e. habits formed over the whole economy average of past consumption (e.g. Augeraud-Veron and Bambi [1]). In

Habit formation has been studied in economic and financial models. The introduction of habits has been crucial to solve the equity premium puzzle, which is one of the most well-known puzzles in finance (i.e. Constantinides [2]);[2] habits were also fundamental in showing that becoming addicted to the consumption of a good can be rational (i.e. Becker and Murphy [3]) and in helping to replicate several stylized facts in macroeconomics, such as those on inflation dynamics.

Interestingly, the existing literature has always focused on formulae for habit formation, which exclude the possibility of finite memory, even though several contributions have argued that this feature seems more consistent with the empirical evidence (e.g. Crawford [4]). The main reason behind this choice is that models with a finite lag structure become much more complicated, and not analytically tractable with the economists' best known methods of optimization. Indeed, the mathematical problem arising in the case of finite memory is a non standard, optimal control problem with delay in the objective functional and in the constraints. A problem of this kind is intrinsically infinite dimensional and has some similarities with recent contributions in the field (see, in particular, the paper of Fabbri and Gozzi [5]), but possesses spe-

this article we study only the case of internal habits and we will often refer to them simply as habits.

[2] The equity premium puzzle refers to the inability of models without habit formation, to explain the differential between the risky rate of return of the stock market and the riskless rate of interest, within reasonable parameter choices.

cific features that do not allow existing theory to be applied, see later in this introduction and Subsection 3.4.

Hence, the aim of this paper is to extend the dynamic programming approach (in particular the results of Fabbri and Gozzi [5]) to internal habit formation models with finite memory, and to analytically solve the Hamilton-Jacobi-Bellman, from now on HJB, equation in order to unveil the dynamics of the optimal paths.

Contribution – Our main contribution is the complete solution, through the dynamic programming approach, of the internal habit formation model with finite memory. By complete solution, we mean that we provide an explicit solution of the associated HJB equation, and we show that this solution is the value function; we also explicitly write a closed loop formula for the optimal strategies.

Although it is possible to study the problem using a modified version of the Pontryagin Maximum Principle (PMP) (see, e.g., Agram et al. [6]), this approach hardly allows the identification of an explicit formulation of the optimal policy (as we do) because of the mixed type equation resulting from the PMP in the presence of retarded control.

It must be noted that the delayed structure of the problem pins down an HJB equation that is a partial differential equation in an infinite dimensional Hilbert space. It is usually impossible to find explicit solutions to this type of equation unless specific assumptions on the production and utility function are introduced. Luckily enough, the linear production function and the homogeneity of the utility function allow an ad hoc approach to explicitly solve the HJB equation and then find the closed loop policy functions.

The dynamic programming approach to optimal control problems with delay has had very few applications in the economic literature. The main reason is probably the intrinsic infinite dimensional structure of such problems: the HJB equation is a Partial Differential Equation in a Hilbert space and the theory for such equations is much less developed than for ones in the finite dimension, due to the lack of local compactness and of the Lebesgue reference measure.

Moreover the known theory for HJB equations in infinite dimension (see, e.g., Li and Yong [7]) does not apply to the typical problems arising in economics for two reasons. First, the presence of state (or state-control) constraints; second, the presence, in the state equation, of first order differential operators (arising in translating the delay equation into an ODE in a Hilbert space, see Subsection 3.3) which do not have regularizing properties.

As far as we know, the first authors to apply the dynamic programming method to such problems were Fabbri and Gozzi [5] in a vintage capital framework, and later Boucekkine et al. [8] and Bambi et al. [9], the latter in a time-to-build model.[3] More recently Boucekkine et al. [14] used it to investigate the compatibility of the optimal population size concepts produced by different social welfare functions and egalitarianism. Most of these papers provided

[3] See. also. [10] for a discrete vs continuous time comparison. Also ~~[11] [12] and [13]~~ for the application of the dynamic programming technique to models with age structure.

explicit solutions to HJB equations, while others (like [12,13]) developed, in special cases, a theory of the existence of regular solutions for them.

Our paper fits into this literature, but contains significant differences in the model and, consequently, in the techniques used to find the solution. The main reason is the presence of the delayed habit formation term in the objective function and, consequently, in the constraints. The down side of this is a lack of regularity of the gradient of the value function (explained in detail in Subsection 3.4, Remark 3.1) which forces us to change the setting and the proofs used in [5], and in all the other quoted papers, because they guaranteed this regularity.

Plan of the paper – The paper is organized as follows. Section 2 presents the general model with habit formation. Section 3, the core, is devoted to the solution of the problem. Section 4 concludes the paper.

2 The Model

Consider a standard neoclassical growth model, where a representative agent maximizes over time the discounted instantaneous utility (here $c(t)$ and $h(t)$ are, respectively, the consumption and the habit at time t):

$$\mathcal{U}(c(t), h(t)) = \begin{cases} \frac{(c(t)-h(t))^{1-\gamma}}{1-\gamma}, \text{ for } c(t) \geq h(t), \\ -\infty, \text{ otherwise}, \end{cases} \tag{1}$$

with $\gamma > 0$ and $\gamma \neq 1$. The instantaneous utility function (1) clearly implies addiction in the habits, since current consumption has to remain higher than the habits over time. The utility function (1) has been widely used in macroeconomics, finance and behavioural economics. In particular it was used in the seminal contribution by Ryder and Heal [15] on habit formation as well as in the article of Constantinides [2] on the solution of the equity premium puzzle. More recent contributions include Augeraud-Veron and Bambi [1].[4] In our paper, habits are formed according to the "general" rule

$$h(t) = \varepsilon \int_{t-\tau}^{t} c(u) e^{\eta(u-t)} du, \qquad \forall t \geq 0, \tag{2}$$

where $\tau > 0$ captures the finite memory effect, $\eta > 0$ measures the persistence of habits, and $\varepsilon > 0$ shows the intensity of habits, i.e. the importance of past consumption relative to current consumption.

Moreover, assume that the representative individual starts with a capital income rk_0 and in each period of time has to decide how much to consume and to save. The interest rate paid on each unit of capital $k(t)$ invested in this riskless technology brings a return (i.e. an interest rate) equal to $r > 0$. Therefore, the optimal control problem, ~~from now on problem~~ $(\mathbf{G})$, to be solved by the representative individual is

$$\begin{aligned}
&\max \int_0^\infty \frac{\left(c(t) - \varepsilon \int_{t-\tau}^{t} c(u) e^{\eta(u-t)} du\right)^{1-\gamma}}{1-\gamma} e^{-\rho t} dt \\
&s.t.\ \forall t \geq 0, \quad \dot{k}(t) = r k(t) - c(t), \\
&\qquad k(t) \geq 0,\ c(t) \geq 0,\ c(t) \geq \varepsilon \int_{t-\tau}^{t} c(u) e^{\eta(u-t)} du, \\
&\qquad k(0) = k_0 > 0, \qquad c(u) = c_0(u)\ \ given\ for\ u \in [-\tau, 0[.
\end{aligned}$$

[4] The case $\gamma = 1$ can be treated exactly as the other ones. Subsection 3.6 explains the main features of this case.

3 Solution of the Internal Habit Formation Model

In this section we solve problem (**G**) by using the dynamic programming approach. As the procedure is quite long we divide it in 6 subsections as follows:

- 3.1. where we give some notations;
- 3.2 where we prove useful results about admissible trajectories and finiteness of the value function;
- 3.3 where we define the equivalent infinite dimensional problem;
- 3.4 where we solve the associated HJB equation explicitly;
- 3.5 where we find the closed loop policy;
- 3.6 where we show how to deal with the case of logarithmic utility.

3.1 Notations

Let us call $\widetilde{c} : [-\tau, \infty[\to \mathbb{R}_+$ the *concatenation* of the initial datum $c_0(\cdot) \in L^1([-\tau, 0[; \mathbb{R}_+)$ and of the control strategy $c(\cdot) \in L^1_{loc}([0, +\infty[; \mathbb{R}_+)$[5], defined formally as

$$\widetilde{c}(s) = \begin{cases} c_0(s) & \text{for } s \in [-\tau, 0[, \\ c(s) & \text{for } s \in [0, \infty[. \end{cases}$$

The state equation

$$\dot{k}(t) = rk(t) - c(t) \tag{3}$$

[5] The space $L^1_{loc}([0, +\infty[; \mathbb{R}_+)$ is the set of all functions from $[0, +\infty[$ to $\mathbb{R}_+$ that are Lebesgue measurable and integrable on all bounded intervals.

has for every $c(\cdot) \in L^1_{loc}([0,+\infty[;\mathbb{R}_+)$, a unique absolutely continuous solution, which will be denoted as $k_{k_0,c(\cdot)}(\cdot)$ and which is given by

$$k(t) = k_0 e^{rt} - \int_0^t e^{r(t-u)} c(u) du. \tag{4}$$

The admissible set of controls is denoted $\mathcal{C}(k_0, c_0(\cdot))$ and is defined as:

$$\mathcal{C}(k_0, c_0(\cdot)) = \Big\{ c(\cdot) \in L^1_{loc}\left([0,+\infty[;\mathbb{R}^+\right) : \; k_{k_0,c(\cdot)}(\cdot) \geq 0$$
$$\text{and } c(t) \geq \varepsilon \int_{t-\tau}^t \widetilde{c}(u) e^{\eta(u-t)} du \geq 0 \text{ for almost every } t \in \mathbb{R}_+ \Big\}.$$

Let us denote $J(k_0, c_0(\cdot); c(\cdot))$ the objective function to maximize, that is

$$J(k_0, c_0(\cdot); c(\cdot)) = \int_0^\infty \frac{\left(c(t) - \varepsilon \int_{-\tau}^0 \widetilde{c}(u+t) e^{\eta u} du\right)^{1-\gamma}}{1-\gamma} e^{-\rho t} dt. \tag{5}$$

We call ($\mathbf{P}$) the problem of finding an optimal control strategy, i.e. a strategy $c^*(\cdot) \in \mathcal{C}(k_0, c_0(\cdot))$ such that

$$J(k_0, c_0(\cdot); c^*(\cdot)) = \sup_{c(\cdot) \in \mathcal{C}(k_0, c_0(\cdot))} J(k_0, c_0(\cdot); c(\cdot))$$

and $-\infty < J(k_0, c_0(\cdot); c^*(\cdot)) < +\infty$. The value function of the problem is defined as

$$V(k_0, c_0(\cdot)) := \sup_{c(\cdot) \in \mathcal{C}(k_0, c_0(\cdot))} J(k_0, c_0(\cdot); c(\cdot)),$$

with $V(k_0, c_0(\cdot)) = -\infty$ if $\mathcal{C}(k_0, c_0(\cdot)) = \emptyset$.

3.2 Admissible paths and finiteness of the value function

The finiteness of the value function V is a preliminary condition to attack the problem with the dynamic programming approach. In this subsection we will establish conditions (namely (12), (13) and(17)) for such finiteness, which will always be assumed in the subsequent subsections. To show that V is finite we has to show, on one side that the set of admissible strategies $\mathcal{C}(k_0, c_0(\cdot))$ is not empty; on the other side that suitable bounds on the objective functional J hold.

The first step to accomplish this task is to provide a lower bound $c^m(\cdot)$ for admissible strategies and, consequently, an upper bound $k^M(\cdot)$ for admissible trajectories (Proposition 3.1). The strategy $c^m(\cdot)$ is the unique solution to the delay equation (6). We then study the behavior of $c^m(\cdot)$ by looking, in Proposition 3.2 at the characteristic equation associated to (6). Such results is the basis to prove Proposition 3.3 which provides conditions under which $\mathcal{C}(k_0, c_0(\cdot))$ is, or is not, empty.

Motivated by the results of Proposition 3.3, in the subsequent discussion, we state conditions (12) and (13) under which we will work, which guarantee $\mathcal{C}(k_0, c_0(\cdot)) \neq \emptyset$. Finally in Proposition 3.4 under conditions (12) and (13) we prove, using suitable bounds on J, that condition (17) guarantees the finiteness of V.

Proposition 3.1 *Fix any initial datum* $(k_0, c_0(\cdot)) \in \mathbb{R}_+ \times L^1([-\tau, 0[; \mathbb{R}_+)$. *Let* $c^m(\cdot) \in L^1_{loc}([0, +\infty[; \mathbb{R}+)$ *be the unique solution of the equation*

$$c^m(t) = \varepsilon \int_{t-\tau}^{t} \widetilde{c}^m(u)\, e^{\eta(u-t)} du. \tag{6}$$

Then any control strategy $c(\cdot) \geq 0$ *satisfying*

$$c(t) \geq \varepsilon \int_{t-\tau}^{t} \widetilde{c}(u) e^{\eta(u-t)} du \tag{7}$$

must also satisfy, for every $t \geq 0$,

$$c(t) \geq c^m(t), \tag{8}$$

Moreover, the state trajectory $k(\cdot)$ *associated to* $c(\cdot)$ *is dominated at any time* $t \geq 0$ *by the solution* $k^M(\cdot)$ *obtained by taking the same initial datum* k_0 *and control* $c^m(\cdot)$

$$k(t) \leq k^M(t) = e^{rt}\left[k_0 - \int_0^t c^m(u)\, e^{-ru} du\right]. \tag{9}$$

Proof First we observe, thanks to standard existence theorems for DDE's (see e.g. Hale and Verduyn Lunel [16], Section 2.2) that equation (6) has a unique solution for every $c_0(\cdot) \in L^1([-\tau, 0[; \mathbb{R}^+)$.

Consider now a control strategy $c(\cdot) \in \mathcal{C}(k_0, c_0(\cdot))$. Constraint (7) together with (6) implies that

$$c(t) - c^m(t) \geq \varepsilon \int_{t-\tau}^{t} [\widetilde{c}(u) - \widetilde{c}^m(u)]\, e^{\eta(u-t)} du, \qquad t \geq 0$$

Clearly, since both functions $c(\cdot)$ and $c^m(\cdot)$ have the same past $c_0(\cdot)$, $\widetilde{c}(t) - \widetilde{c}^m(t) = 0$ for $t \in [-\tau, 0[$. So, calling $c_1(t) := c(t) - c^m(t)$ we get, for $t \in [0, \tau]$,

$$c_1(t) \geq \int_0^t c_1(u) e^{\eta(u-t)} du.$$

This implies, by a simple application of Gronwall inequality (see e.g. [16], Lemma 3.1 p.15), that $c_1(t) \geq 0$ for $t \in [0, \tau]$. For $t \in]\tau, 2\tau]$, the following holds

$$c_1(t) \geq \int_{t-\tau}^{\tau} c_1(u) e^{\eta(u-t)} du + \int_{\tau}^{t} c_1(u) e^{\eta(u-t)} du.$$

Since the function $t \mapsto \int_{t-\tau}^{\tau} c_1(u) e^{\eta(u-t)} du$ is nonnegative for every $t \in]\tau, 2\tau]$, then applying again the Gronwall inequality we get $c_1(t) \geq 0$ for $t \in]\tau, 2\tau]$. We thus prove by induction that $c(t) \geq c^m(t)$. Finally, $k(t) \leq k^M(t)$ is immediate using (8) and formula (4). □

We now look at the properties of the lower bound (for admissible controls) $c^m(\cdot)$, which solves equation (6). The characteristic equation associated with the delay equation (6) (e.g. Hale and Verduyn Lunel [16]) is given by:

$$1 = \varepsilon \int_{-\tau}^{0} e^{(\lambda+\eta)u} du. \tag{10}$$

The location of the characteristic roots $\lambda \in \mathbb{C}$ solving the characteristic equation (10) is given in Proposition 3.2.

Proposition 3.2 *The characteristic equation (10) admits a unique real root we will denote by λ_0. It satisfies $\lambda_0 < \varepsilon - \eta$ and all complex roots solving equation (10) have a real part smaller than λ_0. Moreover,*

- *if $1 - \varepsilon \int_{-\tau}^{0} e^{\eta u} du < 0$, then λ_0 is the only root with a positive real part;*
- *if $1 - \varepsilon \int_{-\tau}^{0} e^{\eta u} du > 0$, all the roots have a negative real part;*
- *if $1 - \varepsilon \int_{-\tau}^{0} e^{\eta u} du = 0$, then $\lambda_0 = 0$ and the other roots have a negative real part.*

Proof First, we study the real roots. Consider the function $\varphi : \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi(\lambda) = 1 - \varepsilon \int_{-\tau}^{0} e^{(\lambda+\eta)u} du.$$

Since $\varphi'(\lambda) = -\varepsilon \int_{-\tau}^{0} u e^{(\lambda+\eta)u} du > 0$, then φ is a strictly increasing function of λ. Moreover,

$$\lim_{\lambda \to -\infty} \varphi(\lambda) = -\infty, \qquad \lim_{\lambda \to +\infty} \varphi(\lambda) = 1$$

and

$$\varphi(0) = 1 - \varepsilon \int_{-\tau}^{0} e^{\eta u} du \geq 1 - \varepsilon\tau, \qquad \varphi(\varepsilon - \eta) = e^{-\varepsilon\tau} > 0.$$

The above equation implies that there exists a unique real root of the equation $\varphi(\lambda) = 0$. Such a root belongs to $]-\infty, \varepsilon - \eta[$.

We now study the location of the complex roots. Taking the real part of equation (10), all complex roots $\lambda = p + iq$ satisfy $1 - \varepsilon \int_{-\tau}^{0} e^{(p+\eta)u} \cos(qu) du = 0$. We get that

$$1 < \varepsilon \int_{-\tau}^{0} e^{(p+\eta)u} du = 1 - \varphi(p).$$

This implies that $\varphi(p) < 0$, and thus $p < \lambda_0$.

Let us now assume that $1 - \varepsilon \int_{-\tau}^{0} e^{\eta u} du < 0$ and consider the function

$$a(\lambda) := (\lambda + \eta)\varphi(\lambda).$$

It can be easily seen that $a(\lambda)$ can be written as $a(\lambda) = \lambda + \eta - \varepsilon \left(1 - e^{-(\lambda+\eta)\tau}\right)$ and that all complex roots of the characteristic equation $\varphi(\lambda) = 0$ are also solutions of $a(\lambda) = 0$. Let us assume that there exists a complex root of $\varphi(\lambda) = 0$, denoted $\lambda = p + iq$ with $p \in]0, \lambda_0[$. Then, we have

$$\mathrm{Re}\,(a(\lambda)) = p + \eta - \varepsilon + \varepsilon e^{-(p+\eta)\tau} \cos(q\tau) < p + \eta - \varepsilon + \varepsilon e^{-(p+\eta)\tau} = a(p) < 0,$$

which contradicts the fact that $\mathrm{Re}\,(a(\lambda)) = 0$. □

Proposition 3.3 gives conditions for admissible controls to exist.

Proposition 3.3 *Fix an initial datum* $(k_0, c_0(\cdot)) \in \mathbb{R}_+ \times L^1([-\tau, 0[; \mathbb{R}_+)$.

(i) $\mathcal{C}(k_0, c_0(\cdot))$ *is nonempty if and only if the control* $c^m(\cdot)$ *introduced in (6) is such that* $k^M(t) \geq 0$ *for every* $t \geq 0$. *If* $c_0(t) > 0$ *on a set of positive*

Lebesgue measure then the above is equivalent to ask that $k^M(t) > 0$ for every $t \geq 0$.

(ii) *In particular, if $\lambda_0 \geq r$ then for any $c_0(\cdot) \in L^1([-\tau, 0[; \mathbb{R}_+)$, such that $c_0(t) > 0$ on a set of positive Lebesgue measure, we have $\mathcal{C}(k_0, c_0(\cdot)) = \emptyset$.*

Proof The first statement is an immediate corollary of Proposition 3.1. In particular, when $c_0(t) > 0$ on a set of positive Lebesgue measure then, by (6) it follows that $c^m(\cdot)$ is always strictly positive, hence $k^M(t) > 0$ for every $t \geq 0$.

Concerning the second statement, we observe that the solution of the equation (6) can be written with a series expansion (see e.g. Corollary 6.4, p.168 of [17]) as follows

$$\tilde{c}^m(t) = \sum_{j=0}^{\infty} p_j(t) e^{\lambda_j t}, \tag{11}$$

where $\{\lambda_j\}_{j \in \mathbb{N}}$ is the sequence of the roots of the characteristic equation (10) and $p_j(t)$ are polynomials of degree less than or equal to $m(j) - 1$ where $m(j)$ is the multiplicity of λ_j. Now, using e.g. Bellman and Cooke [18] (Section 6.7, in particular Theorem 6.5), we can explicitly compute the coefficients of such solutions by using the Laplace transform.

In particular, since λ_0 is a simple root, we have

$$p_0 = \frac{\psi(\lambda_0)}{\varphi'(\lambda_0)},$$

where

$$\varphi(\lambda) = 1 - \varepsilon \int_{-\tau}^{0} e^{(\lambda+\eta)u} du \text{ and } \psi(\lambda) = (1 - \varphi(\lambda)) \int_{-\tau}^{0} c_0(u) e^{-\lambda u} du.$$

Clearly, if $c_0(\cdot) > 0$ on a set of positive Lebesgue measure, we have that $p_0 > 0$ and so the leading term of the series (11) is $p_0 e^{\lambda_0 t}$ and all other terms are complex exponentials with negative real part. So the corresponding state trajectory $k^M(\cdot)$ can be written as

$$k^M(t) = e^{rt}\left[k_0 - \int_0^t p_0 e^{(\lambda_0 - r)u} du + \xi(t)\right],$$

where $\xi(\cdot) : [0, +\infty[\to \mathbb{R}$ is a bounded function coming from the lower order term of the series (11). When $\lambda_0 \neq r$ it follows

$$k^M(t) = e^{rt}\left[k_0 + \frac{p_0}{\lambda_0 - r} + \xi(t)\right] - \frac{p_0}{\lambda_0 - r} e^{\lambda_0 t}.$$

Clearly, when $\lambda_0 > r$ the limit of the above expression is $-\infty$, so the claim follows. When $\lambda_0 = r$ we have

$$k^M(t) = e^{rt}\left[k_0 - p_0 t + \xi(t)\right]$$

and again the limit of the above expression is $-\infty$, so the claim follows. □

Due to the above Proposition 3.3, when $c_0(t) > 0$ on a set of positive Lebesgue measure, we have $\mathcal{C}(k_0, c_0(\cdot))$ nonempty (hence it makes sense to study our problem) if and only if

$$\lambda_0 < r \tag{12}$$

where λ_0 is the unique real root of (10), and

$$k_0 \geq \int_0^{+\infty} e^{-sr} c^m(s) ds, \tag{13}$$

where $c^m(\cdot)$ is the unique solution of (6). We will assume these conditions from now on. Observe that, while (12) only depends on the parameters ε, τ, η, (13) also depends on the initial consumption profile $c_0(\cdot)$ in a nontrivial way: we may say, roughly speaking, that the integral of $c_0(\cdot)$ must be small enough to guarantee that the corresponding $k^M(\cdot)$ is always strictly positive.

Moreover since λ_0 is the highest possible growth rate of the habit, (12) requires that it has to be lower than the interest rate r, which coincides with the maximum growth rate of capital obtainable from the capital accumulation equation when consumption is set to zero. In fact, an economy cannot sustain over time a growth rate that exceeds the real interest rate because capital does not accumulate sufficiently fast to sustain the higher and higher consumption.

Note in particular that (12) is surely true if

$$\varepsilon - \eta \leq r \tag{14}$$

or, since $\frac{\varepsilon}{\eta}(1 - e^{-\eta\tau}) < 1 \Leftrightarrow \lambda_0 < 0$, if

$$\frac{\varepsilon}{\eta}(1 - e^{-\eta\tau}) < 1 \qquad and \qquad r > 0. \tag{15}$$

In the following subsections we will sometimes focus on the case when

$$\lambda_0 < \varepsilon - \eta \leq 0 < r. \tag{16}$$

Indeed the condition $\varepsilon - \eta \leq 0$ is usually assumed in the economic literature (e.g. Constantinides [2]) because it prevents the economy asymptotically converging on the corner solution $c(t) = h(t)$.

Therefore, conditions (12) and (13) are necessary to guarantee that the value function V is not $-\infty$. Here, we give a sufficient condition for the finiteness of V.

Proposition 3.4 *Consider an initial datum* $(k_0, c_0(\cdot)) \in \left(\mathbb{R}_+ \times L^1([-\tau, 0[; \mathbb{R}_+)\right)$. *Assume that (12) and (13) hold true, so* $\mathcal{C}(k_0, c_0(\cdot)) \neq \emptyset$. *If*

$$\rho > r(1-\gamma), \tag{17}$$

then the value function is always finite.

Proof To prove the claim it is enough to prove the following:

(i) If $\gamma \in]0,1[$ then there exists $M_+ > 0$ such that, for all $(k_0, c_0(\cdot))$ in the space $\left(\mathbb{R}_+ \times L^1([-\tau, 0[, \mathbb{R}_+)\right)$,

$$0 \leq V(k_0, c_0) \leq M_+ k_0^{1-\gamma}.$$

(ii) If $\gamma \in]1, +\infty[$ and (15) holds, then there exists $M_- < 0$ such that, for all $(k_0, c_0(\cdot))$ in the space $\left(\mathbb{R}_+ \times L^1([-\tau, 0[, \mathbb{R}_+)\right)$,

$$M_- k_0^{1-\gamma} \leq V(k_0, c_0) \leq 0.$$

We first prove (i). The first inequality is obvious since for $\gamma \in]0,1[$ we always have $J(k_0, c_0(\cdot); c(\cdot)) \geq 0$.

Concerning the other inequality (Fleming and Soner [19], p.30-32, Freni et al. [20]), let us introduce $\zeta(.)$ defined as:

$$\zeta(s) = \int_0^s c(u)^{1-\gamma}\, du$$

and applying Hölder's inequality to $\zeta(s) = \int_0^s s^{1-\gamma} \left(\frac{c(u)}{s}\right)^{1-\gamma} du$ yields

$$\begin{aligned} \zeta(s) &\leq \left(\int_0^s s^{\frac{1-\gamma}{\gamma}} du\right)^{\gamma} \left(\int_0^s \left(\frac{c(u)}{s}\right)^{\frac{1-\gamma}{1-\gamma}} du\right)^{1-\gamma} \\ &\leq s^{\gamma} \left(\int_0^s c(u)\, du\right)^{1-\gamma}, \end{aligned}$$

as $c(u) = rk\,(u) - \dot{k}\,(u)$

$$\int_0^s c\,(u)\,du = \int_0^s rk\,(u)\,du - k\,(s) + k\,(0)\,.$$

Now, according to equation (4), $k\,(s) \leq k\,(0)\,e^{rs}$. Thus, using the fact that $k(s) \geq 0$ for $s \geq 0$ we get

$$\int_0^s c\,(u)\,du \leq k_0 e^{rs} \tag{18}$$

and so

$$\zeta\,(s) \leq s^{\gamma} k_0^{1-\gamma} e^{(1-\gamma)rs}.$$

Now we have

$$J\,(k_0, c_0(\cdot); c(\cdot)) \leq \int_0^{+\infty} \frac{c(s)^{1-\gamma}}{1-\gamma} e^{-\rho s} ds$$

and, integrating by parts and using (18),

$$J\,(k_0, c_0(\cdot); c(\cdot)) \leq \left(\frac{\rho k_0^{1-\gamma}}{1-\gamma} \int_0^{+\infty} s^{\gamma} e^{((1-\gamma)r-\rho)s} ds \right),$$

which proves the claim.

We now prove (ii). The second inequality is obvious since for $\gamma \in]1, +\infty[$ we always have $J\,(k_0, c_0(\cdot); c(\cdot)) \leq 0$.

Concerning the other inequality, we observe that, considering $c^m(\cdot)$ the unique solution of (6) we have, thanks to (13) and (15), that, for $\alpha > 0$ and small enough, the control strategy defined for $t \geq 0$ as $c_1(t) = c^m(t) + \alpha$ is admissible.

Indeed, calling $k_1(\cdot)$ the associated state trajectory, we have

$$k_1(t) = e^{rt} \left[k_0 - \int_0^t e^{-ru} (c^m(u) + \alpha) du \right] =$$

$$= e^{rt}\left[k_0 - \int_0^t e^{-ru}c^m(u)du - \frac{\alpha}{r}\right] + \frac{\alpha}{r}.$$

This term is always positive if

$$\frac{\alpha}{r} \le k_0 - \int_0^{+\infty} e^{-ru}c^m(u)du,$$

which is possible by (13). Moreover the control $c_1(\cdot)$ satisfies the constraint (7) since, substituting it into (6) we get

$$\alpha \ge \alpha\frac{\varepsilon(1-e^{-\eta\tau})}{\eta},$$

which is always true for positive α thanks to (15).

Since $c_1(\cdot)$ is admissible, we have

$$V(k_0, c_0(\cdot)) \ge J(k_0, c_0(\cdot); c_1(\cdot)) \ge \frac{\alpha^{1-\gamma}}{\rho(1-\gamma)}.$$

~~Now it is clear from~~ what was said above ~~that it must be~~ $\alpha \le rk_0$, so the claim follows taking $M_- = \frac{r^{1-\gamma}}{\rho(1-\gamma)}$. □

Observe that condition (17) is the same condition that guarantees bounded utility in the same economic model without habit formation. From now on, we assume that condition (17) holds.

3.3 The Equivalent Infinite Dimensional Problem

Due to the presence of the delay, the optimal control problem is infinite dimensional. We thus need to define a suitable space and an adapted state variable, called structural state, such that, in this space, the structural state equation solves an ODE and such that the objective and constraints can be written without delays. Note that this differs from the previous literature (see e.g. Fabbri and Gozzi [5]), in that the delay does not appear in the state equation. The past of the control strategy appears in the objective functional and in the constraint (7). For this reason the way we choose to rewrite our problem is different from the one given in the previous literature. It may be possible, as proposed in the last part of the introduction, to rewrite the problem differently adding the state variable $h(\cdot)$ defined in (2): this would add another state variable to the system, but would not seem to improve the technical issues that have to be faced to solve the problem, see Remark 3.1.

We work in the Hilbert space $M^2 = \mathbb{R} \times L^2([-\tau, 0[; \mathbb{R})$, with the scalar product defined by

$$\langle (x_0, x_1(\cdot)), (y_0, y_1(\cdot)) \rangle_{M^2} = x_0 y_0 + \int_{-\tau}^{0} x_1(s) y_1(s) \, ds$$

for every $x = (x_0, x_1(\cdot))$ and $y = (y_0, y_1(\cdot))$ in M^2.

As in Vinter and Kwong [21], we now introduce a new state variable. The structural state is defined as follows:

Definition 3.1 Given an initial datum $(k_0, c_0(\cdot)) \in \mathbb{R} \times L^1([-\tau, 0[; \mathbb{R})$, and a control strategy $c(\cdot) \in L^1_{loc}([0, +\infty[; \mathbb{R})$ we define the structural state of our controlled dynamical system at time $t \geq 0$ as the element of M^2:

$$X_{(k_0, c_0(\cdot)), c(\cdot)}(t) = \left(k_{k_0, c(\cdot)}(t), s \mapsto \varepsilon \int_{-\tau}^{s} \tilde{c}(t + u - s) e^{\eta u} du \right). \tag{19}$$

In the following we write $X(t)$ for $X_{(k_0, c_0(\cdot)), c(\cdot)}(t)$ when no confusion is possible. The second component of $X(t)$ is a function of $s \in [-\tau, 0]$ (defined also at $s = 0$, as the integral in (19) makes sense) and we usually write $X_1(t)[s]$ when we want to refer to its value at time t for given $s \in [-\tau, 0]$.

We now characterize the state equation, solved by the state variable.[6] In order to do so, we need to define some operators. We first define the unbounded operator $\mathcal{A}$ on M^2 by

$$D(\mathcal{A}) = \left\{ (x_0, x_1(\cdot)) \in M^2,\ x_1(\cdot) \in W^{1,2}([-\tau, 0]; \mathbb{R}),\ x_1(-\tau) = 0 \right\},$$

$$\mathcal{A}x = (rx_0, -x_1'(\cdot)).$$

Moreover, we define the operators

$$\mathcal{B} : \mathbb{R} \to M^2, \qquad \mathcal{B}\theta = \theta\left(-1, s \mapsto \varepsilon e^{\eta s}\right)$$

and

$$\mathcal{D} : \mathbb{R} \times C([-\tau, 0]; \mathbb{R}) \subset M^2 \to \mathbb{R}, \qquad \mathcal{D}x = x_1(0).$$

Now we show that the structural state satisfies a suitable ODE in the space M^2.

Theorem 3.1 *Given any initial datum* $(k_0, c_0(\cdot)) \in \mathbb{R} \times L^1([-\tau, 0[; \mathbb{R})$ *and any control strategy* $c(\cdot) \in L^1_{loc}([0, \infty); \mathbb{R})$ *the associated structural state is the unique solution of the equation*

$$\begin{cases} \frac{dX(t)}{dt} = \mathcal{A}X(t) + \mathcal{B}c(t), \\ \\ X(0) = \left(k_0, s \mapsto \varepsilon \int_{-\tau}^{s} c_0(u - s)\, e^{\eta u} du\right). \end{cases} \tag{20}$$

[6] In the following we will indicate with $W^{1,2}$ the Sobolev space defined as $W^{1,2} = \{f \in L^2([-\tau, 0]; \mathbb{R}), Df \in L^2([-\tau, 0]; \mathbb{R})\}$.

Proof The proof easily follows by the definition of the structural state and of the operators $\mathcal{A}$ and $\mathcal{B}$. The uniqueness of the solution is similar to Bensoussan et al. ([22], Theorem 5.1, p.282). □

Now we consider the ODE (20) with generic initial datum $x \in M^2$ and call $X(t;x,c(\cdot))$ (or simply $X(t)$ when this is clear from the context) its unique solution for a given control strategy $c(\cdot) \in L^1_{loc}([0,+\infty[;\mathbb{R}_+)$. Taking into account our infinite dimensional setting, we are now ready to reformulate the initial problem (**P**), with the state variable satisfying the state equation (20).

The set of admissible control strategies for a given initial datum in $x \in M^2$ is given by

$$\mathcal{C}_{ad}(x) = \big\{c(\cdot) \in L^1_{loc}([0,\infty[\,;\mathbb{R})\,, \text{ such that}$$

$$X_0(t) \geq 0,\, c(t) \geq 0,\, c(t) \geq X_1(t)[0] = \mathcal{D}X(t) \text{ for all } t\big\}\,.$$

The functional to be maximized becomes

$$J_0(x\,;c(\cdot)) := \int_0^\infty \frac{(c(t) - \mathcal{D}X(t))^{1-\gamma}}{1-\gamma} e^{-\rho t} dt$$

The value function is defined as $V_0(x) := \max_{c(.)\in\mathcal{C}_{ad}(x)} J_0(x\,;c(\cdot))$ where we set $V_0(x) = -\infty$ if $\mathcal{C}_{ad}(x)$ is empty. We now derive the adjoints of operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$.

Lemma 3.1 *The adjoint of $\mathcal{A}$ in M^2 is the operator $\mathcal{A}^* : D(\mathcal{A}^*) \subset M^2 \to M^2$ defined as*

$$\begin{cases} D(\mathcal{A}^*) = \big\{(y_0, y_1(\cdot)) \in M^2 :\, y_1(\cdot) \in W^{1,2}([-\tau,0]\,;\mathbb{R}) \text{ and } y_1(0) = 0\big\}\,, \\ \mathcal{A}^*(y_0, y_1(\cdot)) = (ry_0, s \mapsto y_1'(s))\,. \end{cases}$$

Proof Take $x \in D(\mathcal{A})$ and $y \in M^2$. We have

$$\langle \mathcal{A}x, y \rangle_{M^2} = rx_0y_0 - \int_{-\tau}^{0} x_1'(s)\, y_1(s)\, ds.$$

Using integration by part, it yields to:

$$\langle \mathcal{A}x, y \rangle_{M^2} = rx_0y_0 - x_1(0)\, y_1(0) + x_1(-\tau)\, y_1(-\tau) + \int_{-\tau}^{0} x_1(s)\, y_1'(s)\, ds.$$

As $x \in D(\mathcal{A})$,

$$\langle \mathcal{A}x, y \rangle_{M^2} = rx_0y_0 - x_1(0)\, y_1(0) + \int_{-\tau}^{0} x_1(s)\, y_1'(s)\, ds. \tag{21}$$

To define A^* and $D(\mathcal{A}^*)$, we now use $\langle \mathcal{A}x, y \rangle_{M^2} = \langle x, \mathcal{A}^* y \rangle_{M^2}$, for the $y \in D(\mathcal{A}^*)$ we have to define.

If we denote $\mathcal{A}^* y = (z_0, z_1(.))$,

$$\langle x, \mathcal{A}^* y \rangle_{M^2} = x_0z_0 + \int_{-\tau}^{0} x_1(s)\, z_1(s)\, ds. \tag{22}$$

Comparing equations (21) and (22) yields the definition of $D(\mathcal{A}^*)$ and $\mathcal{A}^* y$ as in the claim. □

Lemma 3.2 *The adjoint of* $\mathcal{B}$ *is*

$$\mathcal{B}^* : M^2 \to \mathbb{R}, \qquad \mathcal{B}^*(y_0, y_1(.)) = -y_0 + \varepsilon \int_{-\tau}^{0} e^{\eta s} y_1(s)\, ds.$$

Moreover the adjoint of $\mathcal{D}$ *is*

$$\mathcal{D}^* : \mathbb{R} \to \mathbb{R} \times [C([-\tau, 0]; \mathbb{R})]^*, \qquad \mathcal{D}^* c = c(0, \delta_0),$$

where δ_0 *is the Dirac's* δ *at point* $t = 0$.

Proof We have

$$\langle \mathcal{B}c, (y_0, y_1(\cdot)) \rangle_{M^2} = c \left(-y_0 + \varepsilon \int_{-\tau}^{0} e^{\eta s} y_1(s)\, ds \right).$$

Moreover

$$\langle \mathcal{D}x, c \rangle_{\mathbb{R}} = c x_1(0) = c\,(0 \cdot x_0 + \delta_0 x_1)$$

and the claim follows. □

3.4 The HJB Equation and its Explicit Solution

The Current Value Hamiltonian, H_{CV}, of our problem is a real valued function defined on a subset of $M^2 \times M^2 \times \mathbb{R}$ called E

$$E = \left\{ (x, p, c) \in D(A) \times M^2 \times \mathbb{R} \right\}$$

and is defined by

$$H_{CV}(x, p\,; c) = \frac{(c - \mathcal{D}x)^{1-\gamma}}{1-\gamma} + \langle \mathcal{A}x, p \rangle_{M^2} + \langle \mathcal{B}^* p, c \rangle_{\mathbb{R}}. \tag{23}$$

When $\gamma > 1$, $H_{CV}(x, p\,; c)$ is not defined at the points where $c = x_1(0)$. At these points, since the utility is $-\infty$, we set $H_{CV}(x, p\,; c) = -\infty$.

Consider now $\mathcal{H}(x, p) = \sup_{c \geq x_1(0)} H_{CV}(x, p\,; c)$, i.e. the maximum value of the Hamiltonian. Note that it may take value $+\infty$, e.g., when $\gamma \in (0, 1)$ and $p = 0$. The HJB equation of the problem solved by the unknown variable v is then

$$\rho v(x) - \mathcal{H}(x, Dv(x)) = 0. \tag{24}$$

Remark 3.1 In the paper [5], and later in [8,9,14], a similar HJB equation, coming from a control problem arising in economics and driven by a delay equation, is solved explicitly. We are now ready to explain the difference between the HJB equations in these previous contribution and (24). The current value Hamiltonian, (23), has three terms. The last one is even better than the one in [5] as the operator B here is bounded, while the one in [5] is unbounded. The second one is exactly the same. The problem comes from the first one, originating from the utility function. Indeed this term here is $\frac{(c-\mathcal{D}x)^{1-\gamma}}{1-\gamma}$ which contains the state variable in the expression $\mathcal{D}x = x_1(0)$. In [5] the analogous term is $\frac{(ax_0-i)^{1-\gamma}}{1-\gamma}$ which contains the first component x_0 of the state variable. This difference is crucial since the presence of the unbounded term $\mathcal{D}x$ in the Hamiltonian forces the candidate solution of the HJB equation (see next Proposition 3.5) to have a gradient not belonging to $D(\mathcal{A}^*)$, while in [5] it does. This fact could also be proved directly from the definition of the value function.

We thus need to depart from the other papers to give a different definition of the solution for our problem. This makes the proofs of our main results (Proposition 3.5, where we prove that v solves the HJB equation, and Propositions 3.6, 3.7 and 3.8, in which we prove that the feedback strategy is admissible and optimal) novel with respect to the existing literature.

We finally notice that rewriting the problem by adding the state variable $h(\cdot)$ defined in (2), would complicate the HJB equation since a new variable would be added. Hence the state space would be $R^2 \times L^2(-\tau, 0)$ and the HJB equation would be more difficult to deal with. On the other hand, the issues described above do not seem to improve with this different setting, since they seem to depend on the structure of the objective function.

Definition 3.2 We say that a function v is a classical solution of the HJB equation (24) in an open set $\mathcal{Y} \subseteq M^2$ if it is differentiable at every $x \in \mathcal{Y}$ and if it satisfies (24) in every point of $\mathcal{Y} \cap D(\mathcal{A})$.

Before finding a solution of the HJB equation, we compute the maximum value Hamiltonian in Lemma 3.3, whose proof is immediate.

Lemma 3.3 *Given any $p \in M^2$ such that $\mathcal{B}^* p < 0$ and any $x \in D(\mathcal{A})$, the function*

$$H_{CV}(x, p\,; \cdot) : [x_1(0), \infty[\to \mathbb{R}$$

admits a unique maximum point

$$c^{\max} = \mathcal{D}x + (-\mathcal{B}^* p)^{-1/\gamma}.$$

So, in this case

$$\mathcal{H}(x, p) = \langle \mathcal{A}x, p \rangle_{M^2} + \frac{\gamma}{1-\gamma} (-\mathcal{B}^* p)^{\frac{\gamma-1}{\gamma}} + \langle \mathcal{D}x, \mathcal{B}^* p \rangle_{\mathbb{R}}.$$

If, on the other hand, $\mathcal{B}^* p \geq 0$, then

$$\sup_{c \geq x_1(0)} H_{CV}(x, p\,; c) = +\infty.$$

We are now going to find an explicit solution of HJB equation (24). Since (24) is analogous to a one-dimensional HJB equation related to a linear problem with a CRRA utility function, we guess that a possible form of the solution is $v(x) = \nu G(x)^{1-\gamma}$, where ν is a constant and $G(.)$ a linear function on M^2.

We next define $G(.)$ and prove in Proposition 3.5 that such a v is indeed a solution. Let us consider, for $x \in M^2$,

$$G(x) = \left(1 - \varepsilon \int_{-\tau}^{0} e^{(r+\eta)s} ds\right) x_0 - \int_{-\tau}^{0} e^{rs} x_1(s)\, ds = \langle x, \kappa \rangle$$
$$\text{where } \kappa = \left(1 - \varepsilon \int_{-\tau}^{0} e^{(r+\eta)s} ds, s \mapsto -e^{rs}\right).$$

It is worth noting that $\kappa > 0$ when we assume (16) i.e. that $r > 0 \geq \varepsilon - \eta$. In fact, looking at κ as function of τ we see that its derivative with respect to τ is always negative. Since it converges to $\frac{r+\eta-\varepsilon}{r+\eta} > 0$ when $\tau \to +\infty$, it must always be positive.

We call $\mathcal{X}$ the open subset of M^2 defined by

$$\mathcal{X} = \{x = (x_0, x_1(\cdot)) \in M^2,\ G(x) > 0\}. \tag{25}$$

Proposition 3.5 *The function* $v(x) = \nu (G(x))^{1-\gamma}$ *with*

$$\nu = \frac{1}{1-\gamma}\left(\frac{\rho - r(1-\gamma)}{\gamma}\right)^{-\gamma}$$

is differentiable for all $x \in \mathcal{X}$ *and is a solution of the HJB equation in* $\mathcal{X}$.

Proof Let $v(x) = \nu (G(x))^{1-\gamma}$ for every $x \in M^2$. Then

$$Dv(x) = (1-\gamma)\nu G(x)^{-\gamma} \kappa.$$

Since $\mathcal{B}^* \kappa = -1$ we have

$$\mathcal{B}^* Dv(x) = (1-\gamma)\nu G(x)^{-\gamma} \mathcal{B}^* \kappa = -(1-\gamma)\nu G(x)^{-\gamma}$$
$$\langle \mathcal{D}x, \mathcal{B}^* Dv(x)\rangle_{\mathbb{R}} = -x_1(0)(1-\gamma)\nu G(x)^{-\gamma}.$$

Now for $x \in D(\mathcal{A})$ we have

$$\langle \mathcal{A}x, Dv(x)\rangle_{M^2} = (1-\gamma)\nu G(x)^{-\gamma} \langle \mathcal{A}x, \kappa\rangle_{M^2}.$$

Moreover, by the definition of $\mathcal{A}$ and κ, we have (integrating by parts and using $x(-\tau) = 0$ since $x \in D(\mathcal{A})$)

$$\begin{aligned}\langle \mathcal{A}x, \kappa\rangle_{M^2} &= r\left(1 - \varepsilon \int_{-\tau}^{0} e^{(r+\eta)s} ds\right) x_0 + \int_{-\tau}^{0} x_1'(s)\, e^{rs} ds \\ &= r\left(1 - \varepsilon \int_{-\tau}^{0} e^{(r+\eta)s} ds\right) x_0 + x_1(0) - r\int_{-\tau}^{0} x_1(s)\, e^{rs} ds \\ &= r\,\langle x, \kappa\rangle_{M^2} + x_1(0)\,.\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{H}(x, Dv(x)) &= (1-\gamma)\,\nu G(x)^{-\gamma}\left[\langle \mathcal{A}x, \kappa\rangle_{M^2} - x_1(0)\right] + \frac{\gamma}{1-\gamma}[(1-\gamma)\nu]^{\frac{\gamma-1}{\gamma}} G(x)^{1-\gamma} \\ &= r\,(1-\gamma)\,\nu G(x)^{1-\gamma} + \frac{\gamma}{1-\gamma}[(1-\gamma)\nu]^{\frac{\gamma-1}{\gamma}} G(x)^{1-\gamma} = \\ &= \nu G(x)^{1-\gamma}\left[r(1-\gamma) + \gamma[(1-\gamma)\nu]^{-\frac{1}{\gamma}}\right].\end{aligned}$$

We can now substitute all the above in the HJB equation getting

$$\begin{aligned}&\rho v(x) - \mathcal{H}(x, Dv(x)) = \\ &= \nu G(x)^{1-\gamma}\left[\rho - r(1-\gamma) - \gamma[(1-\gamma)\nu]^{-\frac{1}{\gamma}}\right]\end{aligned}$$

and the claim follows by the definition of ν. □

The closed loop policy associated with the above solution of the HJB equation (24) is easily found by Lemma 3.3 and is

$$\varphi(x) = x_1(0) + \alpha G(x)\,, \text{ for } x \in \mathcal{X}, \tag{26}$$

where $\alpha = \frac{\rho - r(1-\gamma)}{\gamma}$, which satisfies $\alpha > 0$ thanks to assumption (17).

In the following subsection we prove that the explicit solution of the HJB equation is the value function, and that the closed loop policy gives optimal feedback strategies.

3.5 Closed Loop Policy

We need to determine a set of admissible initial data included in the set $\mathcal{X}$, introduced in (25), such that the candidate optimal feedback φ given in (26) is really optimal. For any x in this set, we then will have that $v(x) = V_0(x)$.

We call $C\left(M^2\right)$ the set of continuous functions from M^2 to $\mathbb{R}$. As in Bambi et al. [9, 23], we give definitions concerning feedback strategies.

Definition 3.3 Given an initial condition $x \in M^2$, we call $\psi \in C\left(M^2\right)$ a feedback strategy related to x if the equation

$$\begin{cases} \frac{dX(t)}{dt} = \mathcal{A}X\left(t\right) + \mathcal{B}\left(\psi\left(X\left(t\right)\right)\right), \\ X\left(0\right) = x \end{cases} \tag{27}$$

has a unique solution $X_\psi\left(t\right)$ in $\Pi = \left\{ f \in C\left([0,\infty[, M^2\right), \frac{df}{dt} \in L^2_{loc}\left([0,\infty), D\left(\mathcal{A}\right)'\right) \right\}$ [7]. The set of feedback strategies related to x is denoted FS_x.

Definition 3.4 Given an initial condition $x \in \mathcal{X}$ and $\psi \in FS_x$, we say that ψ is an admissible strategy if the unique solution $X_\psi(t)$ of (27) satisfies $\psi\left(X_\psi\left(\cdot\right)\right) \in \mathcal{C}_{ad}\left(x\right)$. We denote by AFS_x the set of admissible feedback strategies related to x.

Definition 3.5 We say that ψ is an optimal feedback strategy related to $x \in \mathcal{X}$ if

$$V\left(x\right) = \int_0^\infty \frac{\left(\psi\left(X_\psi(t)\right) - \mathcal{D}X_\psi(t)\right)^{1-\gamma}}{1-\gamma} e^{-\rho t} dt.$$

We denote OFS_x the set of optimal feedback strategies related to x.

[7] We refer to Bambi et al. ([9, 23]) for the definition of Π and ~~for the definition of~~ solutions in Π.

We first prove that our candidate is always in FS_x for all $x \in \mathcal{X}$.

Lemma 3.4 *For every $x \in M^2$ the map*

$$\varphi : M^2 \to \mathbb{R}, \qquad \varphi(x) = x_1(0) + \alpha G(x),$$

is in FS_x.

Proof We have to prove that

$$\begin{cases} \frac{dX(t)}{dt} = \mathcal{A}X(t) + \mathcal{B}(\varphi(X(t))) \\ X(0) = x \end{cases} \tag{28}$$

has a unique solution in Π. Along the trajectories driven by the feedback strategy, we have, using the notation $\widetilde{c}$ and $\widetilde{k}$

$$\begin{cases} \widetilde{c}(t) = \varepsilon \int_{-\tau}^{0} \widetilde{c}(t+u) e^{\eta u} du \\ \qquad +\alpha \left[\left(1 - \varepsilon \int_{-\tau}^{0} e^{(r+\eta)s} ds\right) \widetilde{k}(t) - \int_{-\tau}^{0} e^{rs} \varepsilon \int_{-\tau}^{s} \widetilde{c}(t+u-s) e^{\eta u} du\ ds \right], \\ \dot{\widetilde{k}}(t) = r\widetilde{k}(t) - \widetilde{c}(t), \\ \widetilde{c}(s) = c(s) \text{ for } s \in [-\tau, 0[, \\ \widetilde{c}(0) = \varepsilon \int_{-\tau}^{0} \widetilde{c}(u) e^{\eta u} du \\ \qquad +\alpha \left[\left(1 - \varepsilon \int_{-\tau}^{0} e^{(r+\eta)s} ds\right) k(0) - \int_{-\tau}^{0} e^{rs} \varepsilon \int_{-\tau}^{s} c(u-s) e^{\eta u} du\ ds \right] > 0, \\ \widetilde{k}(0) = k(0). \end{cases} \tag{29}$$

Solutions of the previous system solve $\widehat{k}(t) = k(0)e^{rt} - \int_0^t \widetilde{c}(u) e^{r(t-u)} du$ and

$$\begin{cases} \widetilde{c}(t) = \varepsilon \int_{-\tau}^{0} \widetilde{c}(t+u) e^{\eta u} du \\ \qquad +\alpha \left[\left(1 - \varepsilon \int_{-\tau}^{0} e^{(r+\eta)s} ds\right) \widetilde{k}(t) - \int_{-\tau}^{0} e^{rs} \varepsilon \int_{-\tau}^{s} \widetilde{c}(t+u-s) e^{\eta u} du\ ds \right], \\ \widetilde{c}(s) = c(s) \text{ for } s \in [-\tau, 0[, \\ \widetilde{c}(0) = \varepsilon \int_{-\tau}^{0} \widetilde{c}(u) e^{\eta u} du \\ \qquad +\alpha \left[\left(1 - \varepsilon \int_{-\tau}^{0} e^{(r+\eta)s} ds\right) k(0) - \int_{-\tau}^{0} e^{rs} \varepsilon \int_{-\tau}^{s} c(u-s) e^{\eta u} du\ ds \right] > 0. \end{cases} \tag{30}$$

As the algebraic functional delay equation (30) has a unique continuous solution for $[0, \infty[$ (Bensoussan et al. [22], p.287), system (29) has a unique continuous solution $\left(\widetilde{c}, \widetilde{k}\right)$ on $[0, \infty)$. Denoting $\widetilde{x} = \left(\widetilde{k}, \widetilde{g}\left(t\right)\right)$ where $\widetilde{g}\left(t\right)[s] = \varepsilon \int_{-\tau}^{s} \widetilde{c}\left(t+u-s\right) e^{\eta u} du$, then $\widetilde{x}$ satisfies

$$\begin{cases} \frac{d\widetilde{x}(t)}{dt} = \mathcal{A}\widetilde{x}\left(t\right) + \mathcal{B}\widetilde{c}\left(t\right) \\ \widetilde{x}\left(0\right) = \left(k_0, \widetilde{g}\left(0\right)\right), \end{cases}$$

which has a unique solution using Bensoussan et al. ([22], Theorem 5.1, p.282). Notice that $\widetilde{c}\left(t\right) = \varphi\left(\widetilde{x}\left(t\right)\right)$.

In this way, we have proved existence and uniqueness when the initial datum is of the form $(k_0, \widetilde{g}\left(0\right))$. To get the result for every initial datum $x \in M^2$ we need to set the equation in the space $D(\mathcal{A})'$ and then show that the solution is indeed continuous with values in M^2. This can be done exactly as in Faggian and Gozzi [12], Section 5-6. We do not do it for brevity and since, to solve our starting problem (**P**), we only need to deal with the narrower set of data used here. □

We now want to prove the optimality of φ. This is very difficult to prove (and in general not true) without additional assumptions. So we will prove the optimality of φ when (16) holds and the initial datum x belongs to a given set $I \subset \mathcal{X}$ which includes the data we are interested in. We start by proving a useful invariance property for the trajectory associated to φ.

Proposition 3.6 *For every initial datum $x \in M^2$ the solution $X_\varphi\left(\cdot\right)$ of (28) satisfies*

$$G\left(X_\varphi\left(t\right)\right) = G\left(x\right) e^{\Gamma t}, \ \textit{for all } t \geq 0,$$

where $\Gamma = \frac{1}{\gamma}(r-\rho)$.

Proof It is enough to compute $\frac{d}{dt}G\left(X_{\varphi}\left(t\right)\right)$. Indeed we have

$$\begin{aligned}\frac{d}{dt}G\left(X_{\varphi}\left(t\right)\right) &= \frac{d}{dt}\left\langle X_{\varphi}\left(t\right),\kappa\right\rangle \\ &= \left\langle \mathcal{A}X_{\varphi}\left(t\right)+\mathcal{B}\varphi\left(X_{\varphi}\left(t\right)\right),\kappa\right\rangle.\end{aligned}$$

Now we cannot use existing contributions, such as [5,9,23] to write

$$\left\langle \mathcal{A}X_{\varphi}\left(t\right),\kappa\right\rangle = \left\langle X_{\varphi}\left(t\right),\mathcal{A}^{*}\kappa\right\rangle$$

because κ does not belong to $D(A^{*})$. So we have to compute this term directly.

Since

$$\mathcal{A}\left(x_0,x_1\left(\cdot\right)\right) = \left(rx_0, s\mapsto -\frac{dx_1\left(s\right)}{ds}\right)$$

and

$$\kappa = \left(1-\varepsilon\int_{-\tau}^{0}e^{(r+\eta)s}ds, s\mapsto -e^{rs}\right),$$

integrating by parts as in the proof of Proposition 3.5,

$$\begin{aligned}\left\langle \mathcal{A}X_{\varphi}\left(t\right),\kappa\right\rangle &= rX_{\varphi,0}(t)\left(1-\varepsilon\int_{-\tau}^{0}e^{(r+\eta)s}ds\right)+\int_{-\tau}^{0}\frac{dX_{\varphi,1}(t)[s]}{ds}e^{rs}ds \\ &= X_{\varphi,1}(t)[0]+r\left\langle X_{\varphi}\left(t\right),\kappa\right\rangle = X_{\varphi,1}(t)[0]+rG(X_{\varphi}\left(t\right)).\end{aligned}$$

Moreover, since $\mathcal{B}c = c\left(-1, s\mapsto \varepsilon e^{\eta s}\right)$, and

$$\varphi(X_{\varphi}(t)) = X_{\varphi,1}(t)\left[0\right]+\alpha G\left(X_{\varphi}\left(t\right)\right)$$

then

$$\begin{aligned}&\left\langle \mathcal{B}\varphi\left(X_{\varphi}(t)\right),\kappa\right\rangle = \left\langle \mathcal{B}\left(X_{\varphi,1}(t)\left[0\right]+\alpha G\left(X_{\varphi}\left(t\right)\right)\right),\kappa\right\rangle = \\ &= \left(X_{\varphi,1}(t)\left[0\right]+\alpha G\left(X_{\varphi}\left(t\right)\right)\right)\left(-1+\varepsilon\int_{-\tau}^{0}e^{(r+\eta)s}ds-\varepsilon\int_{-\tau}^{0}e^{(r+\eta)s}ds\right) \\ &= -X_{\varphi,1}(t)\left[0\right]-\alpha G\left(X_{\varphi}\left(t\right)\right).\end{aligned}$$

Hence, summing up, we get

$$\frac{d}{dt} G\left(X_{\varphi}\left(t\right)\right) = \left(r-\alpha\right) G\left(X_{\varphi}\left(t\right)\right).$$

Using that

$$r - \alpha = \Gamma$$

the claim follows. □

Now we define the set I and state a key invariance property.

Proposition 3.7 *The set I defined as*

$$I \boxed{=} \mathcal{X} \cap \left\{ \begin{array}{c} x = (x_0, x_1) \in \mathbb{R} \times W^{1,2}([-\tau, 0]; \mathbb{R}) \subset M^2, \\ \\ x_1\left(s\right) > 0 \textit{ for almost every } s \in]-\tau, 0], \\ \\ x_1\left(-\tau\right) = 0 \end{array} \right\}$$

is invariant for the flow of the autonomous ODE

$$\frac{dX\left(t\right)}{dt} = \mathcal{A}X\left(t\right) + \mathcal{B}\left(\varphi\left(X\left(t\right)\right)\right).$$

Hence, if (16) holds, then for any $x \in I$ we have $\varphi \in AFS_x$.

Proof Let $x = (x_0, x_1(\cdot)) \in I$. We show that the associated solution $X_{\varphi}(t)$ of (28) still belongs to I for every $t > 0$. Since we already know, by Proposition 3.6, that we always have $G(X_{\varphi}(t)) > 0$, it is enough to prove that, for every $t > 0$, $X_{\varphi,1}(t)[\tau] = 0$ and $X_{\varphi,1}(t)[s] > 0$ for almost all $s \in]-\tau, 0]$.

First of all observe that, by using the definition of structural state, we have, for $t \geq 0$ and $s \in [-\tau, 0]$,

$$X_{\varphi,1}\left(t\right)\left[s\right] = \begin{cases} x_1(s-t) + \varepsilon e^{\eta(s-t)} \int_0^t \bar{c}\left(u\right) e^{\eta u} du, & \text{if } t - s - \tau < 0, \\ \\ \varepsilon e^{\eta(s-t)} \int_{t-s-\tau}^t \bar{c}\left(u\right) e^{\eta u} du, & \text{if } t - s - \tau \geq 0, \end{cases} \tag{31}$$

where

$$\bar{c}\,(u) = X_{\varphi,1}(u)\,[0] + \alpha G\,(X_{\varphi}(u)) \quad for \quad 0 \le u < t.$$

According to equation (31), $X_{\varphi,1}\,(t)\,[-\tau] = 0$ for every $t \ge 0$.

Let now $t_0 \ge 0$ be the supremum of all times t such that the above remains true. We are going to prove that $t_0 = +\infty$.

Since $G(X_{\varphi}(t)) > 0$ for every $t \ge 0$, from the above it is clear that, for small $t > 0$ and for every $s \in]-\tau, 0]$ we have $X_{\varphi,1}(t)\,[s] > 0$. So it must be $t_0 > 0$.

Now assume by contradiction that t_0 is finite. Then we have

$$\bar{c}\,(u) = X_{\varphi,1}(u)\,[0] + \alpha G\,(X_{\varphi}(u)) > 0 \quad for \quad 0 \le u < t_0.$$

So according to (31) $X_{\varphi,1}\,(t_0)\,[s]$ satisfies

$$X_{\varphi,1}\,(t_0)\,[s] > 0$$

for every $s \in]-\tau, 0]$. This contradicts the definition of t_0. Thus $t_0 = +\infty$.

Finally, we observe that, if (16) holds, then $X_{\varphi,0}\,(t) > 0$ for all $t \ge 0$. Indeed, since $G\,(X_{\varphi}(t)) > 0$ we have

$$\left(1 - \varepsilon \int_{-\tau}^{0} e^{(r+\eta)s} ds\right) X_{\varphi,0}\,(t) > \int_{-\tau}^{0} e^{rs} X_{\varphi,1}(t)[s] ds \ge 0.$$

Recalling that assumption (16) implies that $1 - \varepsilon \int_{-\tau}^{0} e^{(r+\eta)s} ds > 0$ (see the discussion before Proposition 3.5), we immediately get $X_{\varphi,0}\,(t_0) > 0$. Thus $\varphi \in AFS_x$. $\square$

It now remains to prove that $\varphi \in OFS_x$.

Proposition 3.8 *If $x \in I$ and (16) holds, then $\varphi \in OFS_x$. The associated state-control couple is the unique optimal couple of the problem.*

Proof The first step is to prove that for $x \in I$, we have $v(x) \geq V_0(x)$. To do it, let us consider the solution of the HJB equation $v(x) = \nu (G(x))^{1-\gamma}$ and function $\widetilde{v}(t,x) : \mathbb{R} \times M^2 \to \mathbb{R}$, defined as

$$\widetilde{v}(t,x) = e^{-\rho t} v(x).$$

Now take $x \in I$ and any admissible control $c(\cdot) \in \mathcal{C}_{ad}(x)$. Call $X(\cdot)$ the associated state trajectory starting at x. Noticing that $X(t) \in D(\mathcal{A})$ when $x \in I$, then, we can compute

$$\frac{d\widetilde{v}(t, X(t))}{dt} = -\rho e^{-\rho t} v(X(t)) + e^{-\rho t} < Dv(X(t)), \mathcal{A}X(t) + \mathcal{B}c(t) > .$$

Integrating on $[0,\tau]$ yields to

$$e^{-\rho\tau} v(X(\tau)) - v(X(0)) =$$

$$= \int_0^\tau e^{-\rho t} \left[-\rho v(X(t)) + < Dv(X(t)), \mathcal{A}X(t) > + < \mathcal{B}^* Dv(X(t)), c(t) >\right] dt. \tag{32}$$

Using that $v(X(\tau)) = \nu G(X(\tau))^{1-\gamma}$, we now prove that $\lim_{\tau\to\infty} e^{-\rho\tau} v(X(\tau)) = 0$. Indeed, as

$$G(X(t)) = \left(1 - \varepsilon \int_{-\tau}^0 e^{(r+\eta)s} ds\right) X_0(t) - \int_{-\tau}^0 e^{rs} X_1(t)[s] ds,$$

then we have $G(X(t)) \leq \left(1 - \varepsilon \int_{-\tau}^0 e^{(r+\eta)s} ds\right) X_0(t)$. Moreover, as we have seen in Proposition 3.5 that $1 - \varepsilon \int_{-\tau}^0 e^{(r+\eta)s} ds > 0$, thus

$$e^{-\rho\tau} G(X(\tau))^{1-\gamma} \leq \left(1 - \varepsilon \int_{-\tau}^0 e^{(r+\eta)s} ds\right)^{1-\gamma} e^{-(\rho-(1-\gamma)r)\tau} \left(\frac{X_0(t)}{e^{r\tau}}\right)^{1-\gamma}.$$

According to Proposition 3.1, $X_0(t) \leq k^M(t)$ whose growth rate is bounded by r. We thus have that

$$\lim_{\tau \to \infty} e^{-\rho\tau} v\left(X\left(\tau\right)\right) = 0.$$

Hence, using that $x = X\left(0\right)$ and taking the limit as τ tends to infinite in (32), we obtain

$$-v\left(x\right) = \int_0^{+\infty} e^{-\rho t}\left[-\rho v\left(X\left(t\right)\right) + < Dv\left(X\left(t\right)\right), \mathcal{A}X\left(t\right) > + < \mathcal{B}^* Dv\left(X\left(t\right)\right), c\left(t\right) >\right] dt. \tag{33}$$

Using the definition (23) of the current value Hamiltonian and definition (5) of the objective function, we get

$$v\left(x\right) - J_0\left(x; c(\cdot)\right) = \int_0^{\infty} e^{-\rho t}\left(\rho v\left(X\left(t\right)\right) - H_{CV}\left(X\left(t\right), Dv\left(X\left(t\right)\right), c(t)\right)\right) dt.$$

As the value function solves $\rho v\left(x\right) - \mathcal{H}\left(x, Dv\left(x\right)\right) = 0$, the above implies that

$$v\left(x\right) - J_0\left(x; c(\cdot)\right) = \int_0^{\infty} e^{-\rho t}\left[\mathcal{H}\left(X\left(t\right), Dv\left(X\left(t\right)\right)\right) - H_{CV}\left(X\left(t\right), Dv\left(X\left(t\right)\right), c(t)\right)\right] dt. \tag{34}$$

According to the definition of $\mathcal{H}$, for every admissible control the integrand of the above right hand side is always non-negative. This implies, according to the definition of V_0, that

$$v\left(x\right) \geq V_0\left(x\right)$$

and this must be true for every $x \in I$.

Moreover, choosing $c(t) = \varphi(X_\varphi(t))$ (which is admissible thanks to Proposition 3.7) clearly makes the right hand side become zero, making this control strategy is optimal. This implies that $v\left(x\right) = V_0\left(x\right)$ for every $x \in I$.

Finally, if $c^1(\cdot)$ is another optimal strategy (with associated state trajectory $X^1(\cdot)$), it must satisfy (34) (where now $v = V_0$ since they are equal on I). So, for a.e $t \geq 0$,

$$\mathcal{H}\left(X^1(t), Dv\left(X^1(t)\right)\right) - H_{CV}\left(X^1(t), Dv\left(X^1(t)\right), c^1(t)\right) = 0$$

which implies, according to lemma 3.3 that, for almost every t, $c^1(t) = \varphi(X^1(t))$. By the uniqueness of the solutions of the closed loop equation (28), proved in Lemma 3.4, we get, t a.e., $c^1(t) = c(t)$. □

It is worth noting that the optimality of φ depends on the initial datum x belonging to the set I; this implies a restriction on the initial value of capital that we may choose. This restriction will be made explicit in the next Proposition and its economic meaning will be also explained.

Proposition 3.9 *Given any initial datum* $(k_0, c_0(\cdot))$*, the problem* ***(P)*** *has a unique optimal state-control couple* $(k^*(\cdot), c^*(\cdot))$. *Such a couple is the only one that satisfies the closed-loop formula:*

$$\frac{c(t) - h(t)}{r - \Gamma} =$$
$$= \left(1 - \varepsilon \int_{-\tau}^{0} e^{(r+\eta)s} ds\right) k(t) - \left[\frac{h(t)}{r+\eta} - \varepsilon e^{-(r+\eta)\tau} \int_{t-\tau}^{t} \frac{e^{r(t-s)}}{r+\eta} \tilde{c}(s)\ ds\right]$$

where $h(t)$ *is given by the equation*

$$h(t) = \varepsilon \int_{t-\tau}^{t} \widetilde{c}(u) e^{\eta(u-t)} du \qquad \forall t \geq 0. \tag{35}$$

Proof We have, by the definition of the optimal feedback map φ, that, on the optimal path,

$$c(t) - h(t) = \alpha G(X(t))$$
$$= (r - \Gamma)\left[\left(1 - \varepsilon \int_{-\tau}^{0} e^{(r+\eta)s} ds\right) X_0(t) - \int_{-\tau}^{0} e^{rs} X_1(t)[s] ds\right].$$

We know that $X_0(t) = k(t)$, while $X_1(t)[s] = \varepsilon \int_{-\tau}^{s} \tilde{c}(t+u-s)e^{\eta u}du$ so, substituting, we have,

$$\frac{c(t)-h(t)}{r-\Gamma} = \left(1-\varepsilon\int_{-\tau}^{0} e^{(r+\eta)s}ds\right)k(t) - \varepsilon\int_{-\tau}^{0} e^{rs}\int_{-\tau}^{s}\tilde{c}(t+u-s)e^{\eta u}duds.$$

We integrate by parts and, with straightforward computations, we obtain

$$\int_{-\tau}^{0} e^{rs}\int_{-\tau}^{s}\tilde{c}(t+u-s)e^{\eta u}duds$$
$$= \frac{1}{r+\eta}\int_{-\tau}^{0}\tilde{c}(t+v)e^{\eta v}dv - e^{-(r+\eta)\tau}\int_{t-\tau}^{t}\frac{e^{r(t-s)}}{r+\eta}\tilde{c}(s)\ ds$$

which proves the claim. □

Using the above result, it is not difficult to prove, by straightforward computations, that given any initial data $(k_0, c_0(\cdot))$, there exists a Λ such that, along an optimal trajectory, the optimal control $c(\cdot)^*$ satisfies $c(t)-h(t) = \Lambda e^{\Gamma t}$ with

$$\frac{\Lambda}{(r-\Gamma)} = \left(1-\varepsilon\int_{-\tau}^{0} e^{(r+\eta)s}ds\right)k(0) - \left[\frac{h(0)}{r+\eta} - \varepsilon e^{-(r+\eta)\tau}\int_{-\tau}^{0}\frac{e^{-(r)s}}{r+\eta}c(s)\ ds\right]. \tag{36}$$

It is worth noting that the constraint, $c(t) \geq h(t)$, is respected if $\Lambda > 0$ or, in terms of the initial capital stock, if

$$k(0) \geq \frac{h(0)}{r+\eta-\varepsilon+\varepsilon e^{-(r+\eta)\tau}} - \frac{\varepsilon e^{-(r+\eta)\tau}}{r+\eta-\varepsilon+\varepsilon e^{-(r+\eta)\tau}}\int_{-\tau}^{0} e^{-ru}c(u)du.$$

In the specific case, $\tau = \infty$ and $\varepsilon = \eta$ this condition becomes $rk(0) > h(0)$ meaning that capital income (which, in our context, coincides with the initial wealth) has to be higher than the initial habits, otherwise an initial consumption higher than $h(0)$ will pin down a consumption path that is not sustainable over time, since it is financed with the resources coming from disinvestments.

In the case with a finite τ, this condition becomes less restrictive as the first term on the right hand side of the inequality becomes smaller and the second negative term appears. The reason being that the stock of habits is now formed over a finite consumption history and, therefore, less resources are needed at the beginning because the past consumption affecting the habit formation will be completely "depreciated" after a period of length τ.

3.6 The Case of Log Utility

We sketch here how the results proved in the previous subsections can be adapted to cover the case when $\gamma = 1$, i.e. when the utility is logarithmic. In this case we differ from (1), having

$$\mathcal{U}(c(t), h(t)) = \begin{cases} \log(c(t) - h(t)), \text{ for } c(t) \geq h(t), \\ -\infty, \text{ otherwise.} \end{cases} \tag{37}$$

The set of admissible strategies and the value function are defined as in Subsection 3.1.

The results of Subsection 3.2 still hold. Propositions 3 and 4 are exactly the same as they do not depend on the objective functional. Proposition 5 can be proved by putting $\gamma = 1$ in its statement. The method of proof is a bit different and follows the arguments of [20] to find the required estimates.

The material of Subsection 3.3 is exactly the same as it is not affected by the choice of $\mathcal{U}$.

Concerning Section 3.4 the current value Hamiltonian is now

$$H_{CV}(x, p\,; c) = \log(c - \mathcal{D}x) + \langle \mathcal{A}x, p \rangle_{M^2} + \langle \mathcal{B}^* p, c \rangle_{\mathbb{R}} \tag{38}$$

while the maximum value of the Hamiltonian is still defined by

$$\mathcal{H}(x,p) = \sup_{c \geq x_1(0)} H_{CV}(x,p\,;c)$$

and it can take the value $+\infty$, e.g. when $p = 0$.

The HJB equation of the problem solved by the unknown variable v is like (24), and Lemma 12 is satisfied with

$$c^{\max} = \mathcal{D}x + (-\mathcal{B}^* p)^{-1}\,.$$

Hence, in this case

$$\mathcal{H}(x,p) = \langle \mathcal{A}x, p\rangle_{M^2} - \log(-\mathcal{B}^* p) - 1 + \langle \mathcal{D}x, \mathcal{B}^* p\rangle_{\mathbb{R}}\,.$$

Proposition 13 holds, with substantially the same proof, with

$$v(x) = \frac{1}{\rho}(\log G(x) + a)$$

where $a := \frac{r}{\rho} - 1 + \log\rho$. The candidate optimal feedback map φ is exactly as in (28) with $\gamma = 1$, hence $\alpha = \rho$.

Concerning Section 3.5, Lemma 17 is exactly the same, while Propositions 18 and 19 still hold with $\gamma = 1$, and the same proof. Proposition 20 still holds but, as with Proposition 5, it uses a different proof which follows the arguments of [20] to prove the required estimates. Finally Proposition 21 holds in exactly the same way.

4 Conclusion

In this paper we have solved a model with habits when their formation is described by a general functional form that allows for finite memory. To this extent, we have generalized the definition of habits used in the literature. Most importantly, we have provided the methodological background to deal with optimal control problems, with distributed delay in the objective and in the constraints, which extends the tools provided by the existing literature on dynamic programming with delay.

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Supplementary Material

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Answer to the Associate Editor

Dear Editor

we thank you for the opportunity to revise the paper. In the revised version, we have answered to the last questions from the two referees.

In addition we have used the JOTA template, followed the editing recommendations, and also checked the English.

Concerning the issues raised by the first referee:

1. We have changed the way we denote a real number.
2. We have changed γ with g.
3. All the misprints have been corrected.

Concerning the issues raised by the second referee:

1. We rewrote part of Section 3.2. We agree with the referee that, there, the main ideas were not well explained. For this reason we added an explanation of the main results of the subsection before Proposition 3.1, we have modified slightly the statement of Proposition 3.1 and 3.3, and, finally, we have rewritten the discussion after Proposition 3.3. Moreover, for the reader convenience, we also changed the title of the subsection and added a short summary of the whole Section 3 just before the beginning of Subsection 3.1.
2. The English has been checked carefully.